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Le fascicule D du Guide pédagogique en mathématique vient appuyer le programme proposé pour l'enseignement primaire en reprenant, dans un contexte plus particulier et davantage appliqué, les objectifs énoncés dans ce programme. C'est un instrument de travail destiné à soutenir l'action des enseignants et des autres personnes intéressées à l'apprentissage des mathématiques. On y retrouve des suggestions, des recommandations, des commentaires, des affirmations, des exemples, des illustrations. Le guide comprend trois chapitres: 1. les fondements de l'apprentissage des entiers rationnels; 2. l'addition des entiers rationnels; 3. la soustraction des entiers rationnels.

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ELEMENTARY SCHOOL

MATHEMATICS

BOOKLET D
RELATIVE INTEGERS

Québec

EIUQ 8870363

Curriculum Guide

ELEMENTARY SCHOOL

MATHEMATICS

BOOKLET D
RELATIVE INTEGERS

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Direction de la formation générale

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INTRODUCTION

The use of fractions and negative numbers often causes several difficulties. For example, it is not possible to buy 2,5 radios with $ 75,00 if the price of one is $ 30,00. On the other hand, with $ 0,75 it is possible to buy 2,5 metres of electrical wire at $ 0,30 per metre. There are, therefore, situations in which the use of fractions and negative numbers is questionable. Similarly, there are contexts in which it is possible to speak of numbers smaller than zero, and others in which it is not. A woman might not have any children but may not have fewer than zero. A box might be empty, but it may not contain less than nothing. Certain situations nevertheless are particularly suited to the use of negative integers.

When it is sufficiently cold, the thermometer may register gradations of less than zero. A temperature of -10°C means that the temperature of the surrounding air is 10 degrees below zero. Likewise, it is possible to speak of an object, which, thrown to a height of 50 metres, passes in its descent through the 40, 30, 20, 10, and 0 metre points. If this same object is thrown from an airplane over the ocean, it will, if it is heavier than water, continue to fall after having touched the surface of the water; it could even touch bottom at a "height" of -15 m. Finally, if an automobile backs up at a speed of 10 km per hour, it could be said that this automobile "advances" at -10 km/h. The minus reverses everything, something like the phenomenon of the reflection of trees and houses in the calm water of a lake.

For a long time, mathematicians hesitated to use negative numbers, but eventually it was realized that these numbers could be added and subtracted, multiplied and divided. The use of relative integers finally came to be accepted when it became apparent that the system of numbers formed by the addition of the positive integers and of zero presented a reasonable and coherent mathematical model capable of solving a multitude of physical problems.

In secondary school, relative integers have been taught and used for a long time. More recently, there has been a tendency to introduce the study of this group of numbers in elementary school.

Why begin the study of relative integers in elementary school? Does the environment in which the child develops justify the introduction of this system of numbers, and if so, to what extent? To what extent should this instruction be included in the objectives for elementary school?

Strictly speaking, it would certainly be possible to dispense with relative integers at this level, just as the concepts relating to sets, symmetries, relations, logic blocks, "topology" and many other things could be avoided. But the fact remains that the teaching of mathematics has evolved considerably through trial and error. If a certain amount of progress has been made in this domain, it is without doubt due to the examination and careful consideration of new directions. Relative integers belong to that category of subjects which have been the focus of study and experimentation in the recent past.

Each time we find a point or place on an axis in relation to a point of reference on this axis, we use a model borrowed in some way from the model of relative integers. It is the same when speaking of "2512 Lamontagne Street West," of "1035 Lenoir Street East," of "381 Saint-Louis North" or of "638 Saint-Jacques South". It is also the case with expressions such as "before Québec" or "after Québec," "above" or "below zero," "he is 'in the red,'" and the terms "to the left," "to the right," "in front of," "behind," etc. which take the place of the signs "+" and "-" in the traditional writing of relative integers.

In this context, relative integers are a part of the child's world, and should, therefore be the subject of suitable teaching. In the following pages an attempt is made to answer the questions which have just been raised.

Chapter 1

Foundations for the Learning of Relative Integers

1.1 Mathematical Approaches

During the learning process, children quickly perceive that subtraction does not always result in an answer in the set of natural numbers. (See Column II, below.)

The need to invent numbers which could express the results of these operations and appear as an extension of the set of natural numbers stems from this:

Examine the following equations:

I

$7 = 5 + \square$
$8 = 6 + \square$
$9 = 7 + \square$
$10 = 8 + \square$
.....

The answer, which is the same for all of these equations, is:

$\boxed{2}$

This same answer could be expressed by any ordered pair appearing in the equations above, even if these ordered pairs are not very explicit:

(7, 5)
(8, 6)
(9, 7)
(10, 8)

II

$5 = 7 + \square$
$6 = 8 + \square$
$7 = 9 + \square$
$8 = 10 + \square$
.....

Here, as in the first instance, the answer is the same in each case:

$\boxed{-2}$

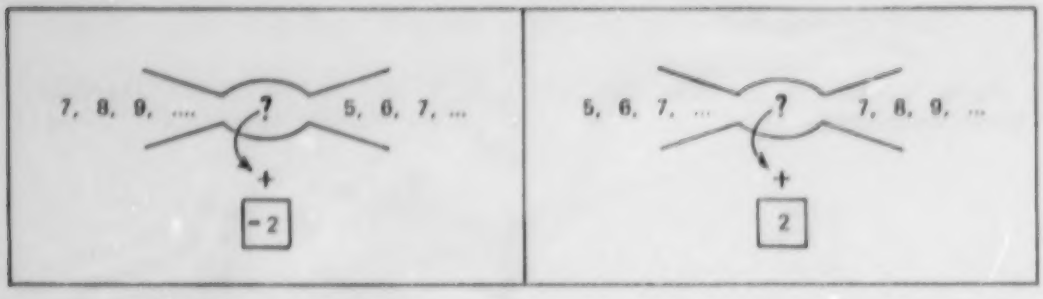
It could be expressed by any of the following ordered pairs from the equations above:

(5, 7)
(6, 8)
(7, 9)
(8, 10)

It can be observed that, in the left-hand column, the first element in each ordered pair is 2 units higher than the second element, while in the right-hand column, just the opposite occurs; that is, the first element of each ordered pair is 2 units less than the second element. Consequently, it could be said that:

2 is the name of (2, 0), (3, 1), (4, 2), ...;
3 is the name of (3, 0), (4, 1), (5, 2), ...;
-2 is the name of (0, 2), (1, 3), (2, 4), ...;
0 is the name of (0, 0), (1, 1), (2, 2), ...;
etc.

Analogous results could have been obtained with a function machine.



Here, it is evident that, starting with the natural numbers, it is possible to invent other numbers. The study of dominance, which has appeared recently, is part of this type of approach which is characterized particularly by the comparison of sets, measures, rods, black or white balls, etc. The concept of dominance will be referred to later in this chapter in connection with manipulations and games, which should make it easier to understand.

1.2 Intuitive Approach

Relative integers are often introduced by means of a number line, directed displacement (arrows), the rungs of a ladder, graduations on a straight line or on a thermometer, the concepts of debit and credit, of profits and losses, or any other application of this sort. (See Section 1.3, which deals with ordering on a straight line and with plotting.) However, this process is not without danger. Even though the examples mentioned are suitable areas of application for the set of numbers considered here, it cannot be denied that these same numbers present serious problems of interpretation and utilization in elementary school. Certain accompanying concepts are particularly difficult, notably those of gains, displacements, or negative time.[1] It would be pointless to dwell upon the difficulty experienced by children of this age in interpreting such expressions.

In fact, it is quite difficult to find a physical model which adequately illustrates relative numbers and operations on these numbers at the same time. The model of the number line is the one which is most frequently used for this purpose.

1.3 Locating a Point on a Straight Line

One of the most frequent applications of a set $\mathbb{Z}$ is locating points on a straight line. When exactly must relative integers be used in place of natural numbers in locating points on a straight line? A straightforward,

---

1 See p. 1: - "height" of -15 m,
- automobile which "advances" at -10 km/h.

but somewhat naive answer is, each time the situation requires it. This is the case, for example, with the scale of the thermometer, with certain movements, with the water level, with the unfolding of an event in relation to time, etc. In each of these cases, locating points on a straight line implies that these points may be located on either side of a certain point taken as a point of reference, that is to say, the "zero" of the set $\mathbb{Z}$ (See the Introduction, p. 1).

Accordingly, each coordinate of a point on a straight line must possess two characteristics: it must indicate the distance separating this point from a point of reference and indicate the direction of this point in relation to this point of reference.

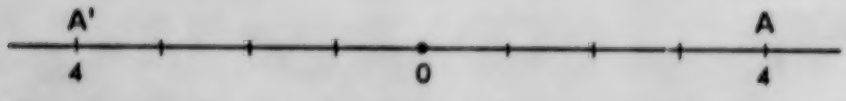


FIGURE 2

Thus, on the straight line in Figure 2, the points A and A' are both designated by the number 4. From this arises the need to imagine a way to represent not only the distance between a point and a point of reference, but also the direction of this point in relation to the point of reference: to the right or the left, above or below. Each of the the symbols used to locate a point on a straight line, therefore, must indicate these two elements. It could be agreed that the numbers located above or to the right of the point of reference ("zero") are to be written in <u>black</u>, while those which are located below or to the left of the point of reference are to be written in <u>red</u>.

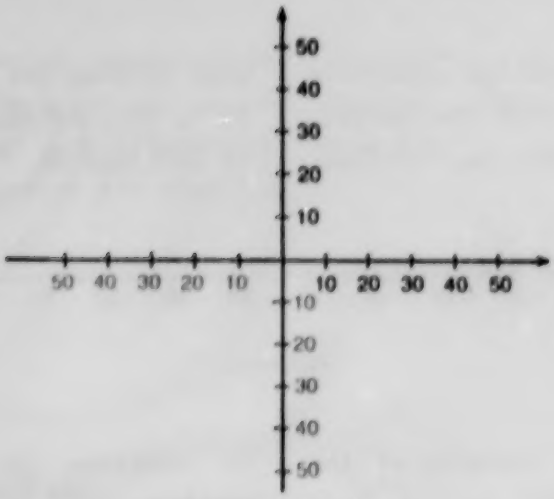


FIGURE 3

Obviously, the choice of red is completely arbitrary and could in fact be replaced by any other equally arbitrary choice: Roman numerals, underlined digits: 4, 5, digits encased in boxes: 6, 9, etc. The convention is that the mathematical signs "+" and "-" be used, the "+" sign for the numbers located to the right or above the "zero" and the "-" sign in the opposite cases. Some confusion can arise from this convention, since the same signs are used to represent the operations of addition and subtraction. For this reason, at the initial stages of learning, the following writing is suggested,

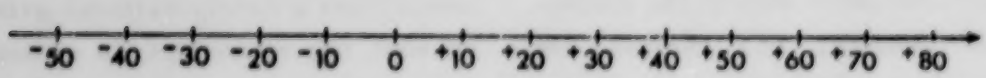


FIGURE 4

where the "+" and "-" signs are placed above and to the left of the number, instead of in the middle. Others nevertheless prefer to adopt

the conventional notation concerning the use of signs from the outset, going so far as to omit the "+" sign when writing the positive integers. It is this approach which is adopted in this booklet.

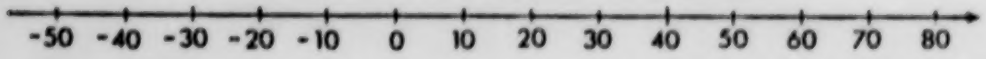


FIGURE 5

In addition to the examples of the thermometer and the falling body cited in the introduction, the countdown for a rocket launch could also be mentioned:

-40, ..., -30, ... -20, ..., -3, -2, -1, 0

The social studies and natural sciences will probably furnish the most applications of a plotting system. Examples: Twenty kilometres west of Montreal, fifteen kilometres north of St-Jérôme, 523 Archambault Street East, 1603 Lapierre Boulevard South, seventy-five years before the discovery of America, one hundred and twenty years after the fall of the Roman Empire, the year thirty-five before Christ, etc. (See the Introduction, p. 1.)

In addition to the problem of plotting there is the problem of the location of points in relation to each other. Children should realize that -10°C is colder than -4°C, while 10°C is warmer than 5°C, that 425 Notre-Dame Street West is to the east of 1250 West on the same street and to the west of 125 East. Yet it should be noted that in the first case, the expressions "colder," "warmer," "not as cold as," and "not as warm as" sometimes make the mathematical representations difficult. The unwieldiness of the language used in the second example constitutes a fairly faithful example of what one encounters in mathematical language when it is said, for example, that -5 is not as big as -3 and bigger than -7, or that 5 is not as big as 7 and bigger than 3.

Since most problems of communication are problems of language, it should not be surprising that mathematics attempts to standardize and simplify expression of this mathematical concept. Thus, we would write, for example:

(a) $-10 < .4$
(b) $10 > 5$
(c) $.7 > .5 < 2$

1.4 Plotting on a Plane

From determining location on a straight line, it is relatively easy to proceed to plotting on a plane. Statistical tables, road maps, city street maps, and geographical maps are among the more common examples of the type of plotting on a plane a child can practice. Even if the child does not generally need to use relative integers for the activities, he still must make use of two coordinates to locate a point on the any plane. (See Booklet F, Chapter 6: Plotting, Coordinates, and Graphs.)

The case of geographical coordinates is, however, a rather special case among the examples mentioned above. Actually, the geographical plane is a curve plane in which circles are used instead of the straight lines of a Cartesian plane.

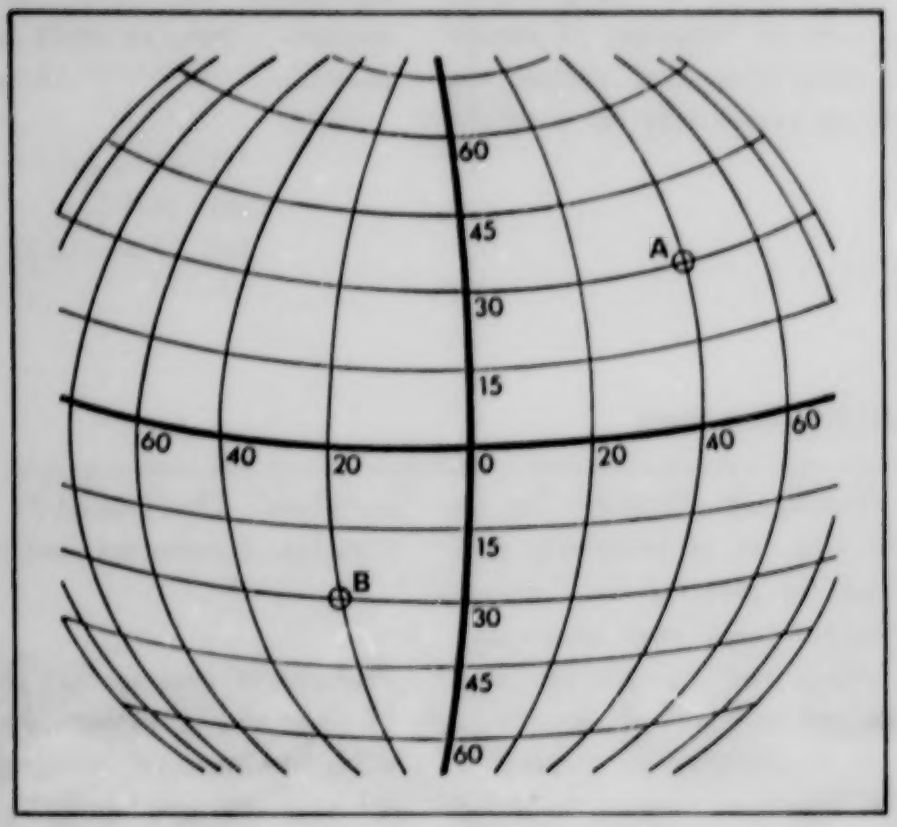


FIGURE 6

Figure 7 is more or less the kind of image that could be found on a Cartesian plane, where east and north are replaced with the "+" sign and west and south are replaced with the "-" sign. Thus, in Figure 6, the coordinates of point A are: longitude 40° east, latitude 30° north, and those of point B are: longitude 20° west, latitude 30° south.

To plot points on a Cartesian plane in reference to fixed axes, coordinates expressed in terms of relative integers (See Figure 7) will be used.

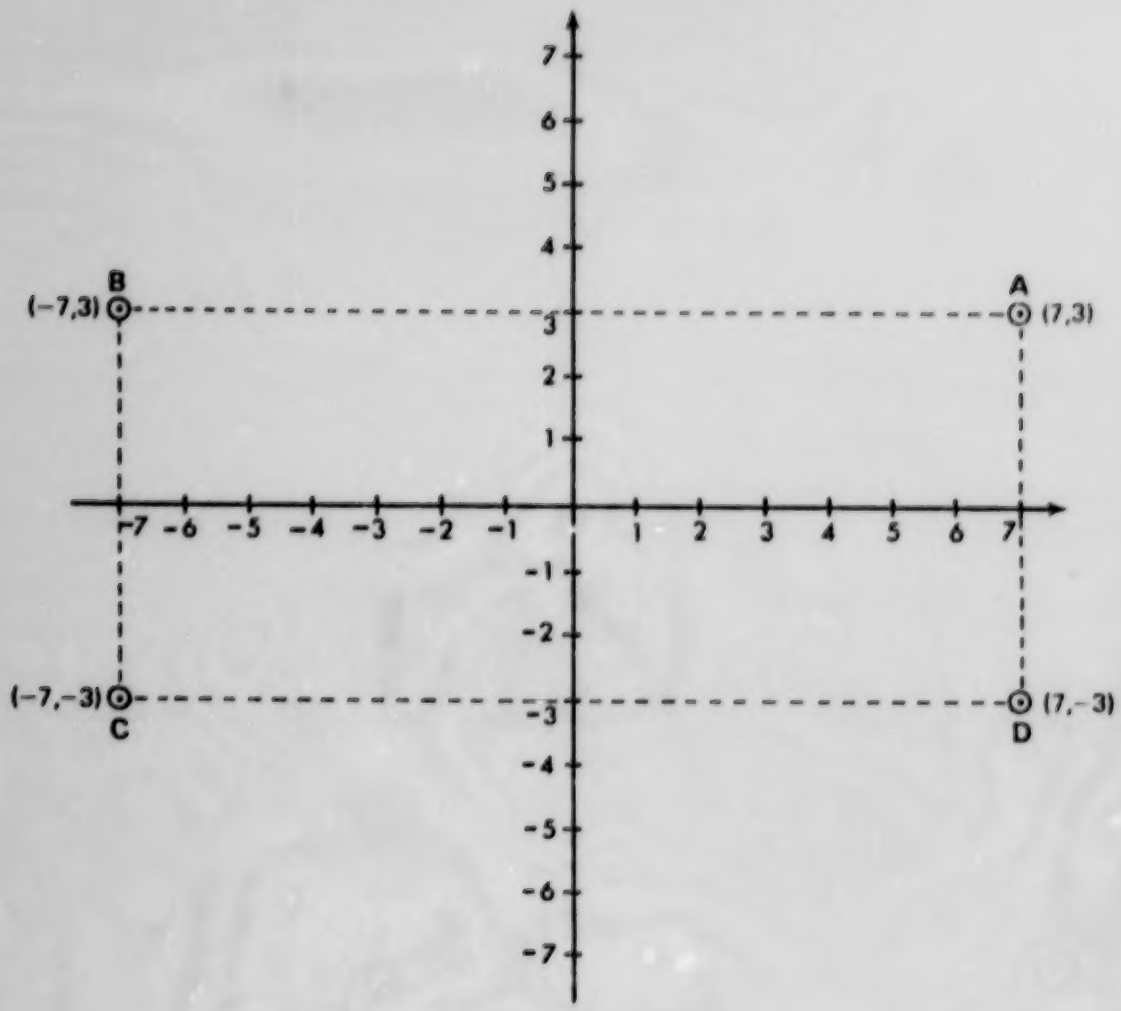


FIGURE 7

1.5 Other Applications Concerning Relative Integers

In the light of the preceding paragraphs, it should not be concluded that plotting on a straight line or on a graph is the principal or only application of relative integers. Even if their applications are limited in elementary school, they can be used to mathematize certain manipulations and even games (See the last paragraph of Section 1.1, but refer particularly to Chapters 2 and 3).

Chapter 2

Addition of Relative Integers

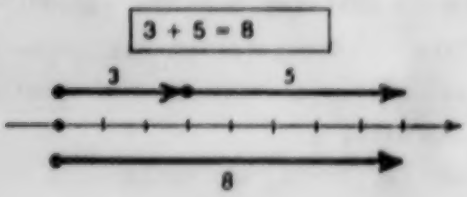
As mentioned in Chapter 1, the approach used in learning relative integers can vary. Use can be made of various types of exploration activities, which should be seen as complementary rather than mutually exclusive. Generally these activities are used to illustrate the important concept of <u>additive inverse</u>. It is in this context of exploration activities that the learning of relative integers should take place in elementary school.

Here are some exploration models which could facilitate the learning of relative integers.

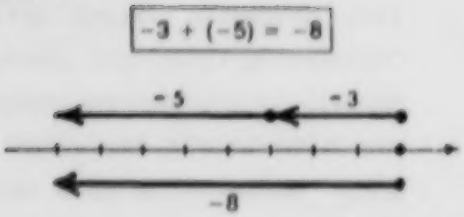
2.1 Displacement on a Straight Line

In the following examples, the addition of a positive number is represented by a displacement to the right, while the addition of a negative number is represented by a displacement to the left.

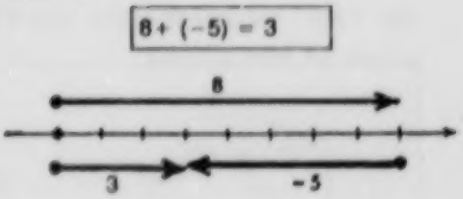
EXAMPLE 1:



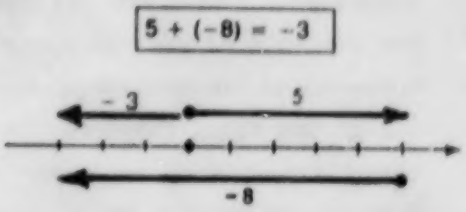
EXAMPLE 2:



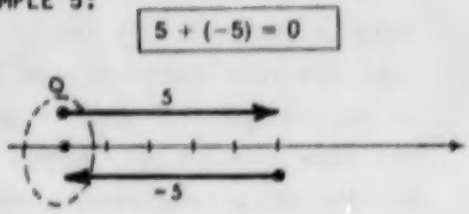
EXAMPLE 3:



EXAMPLE 4:



EXAMPLE 5:



In this type of exercise two successive movements (black arrows) are replaced by a single movement (red arrow). Graphic analysis of the operation should emphasize that the arrow which represents the second movement (the second term of the addition) must always begin where the first arrow ends. The resulting arrow must join the starting point of the first movement with the finishing point of the second.

In Example 5 above it is clear that there cannot be an arrow to illustrate the result of the addition since the result of the two successive movements is a return to the starting point. This case illustrates the property of additive inverse.

2.2 Using Ordered Pairs of Natural Numbers

Different procedures or contexts are suggested. These are all centred on the concepts of dominance and equivalence class as presented in Section 1.1 of the first chapter. In each of these situations, in order to find the sum of ordered pairs, the sum of the first elements must be found, and then that of the second elements, and a new ordered pair formed with the results obtained. It should be noted that the same number may be added to or subtracted from both elements of the same ordered pair without changing its value (equivalence class) (See the note at the end of Section 2.2.1).

2.2.1 Black Marbles and White Marbles

The use of different-coloured marbles provides an excellent means to compare two groups of marbles and to determine how many more or less one group contains than the other. This type of manipulation has given rise to what is termed the dominance approach.

Here is an example:

Silvio has three black marbles and one white one (3, 1), and Joann has one black and four white (1, 4). Silvio has a surplus (dominance) of 2 black marbles while Joann has a dominance of -3 since three more black marbles are needed to equal the number of white ones. If Joann and Silvio decide to put all of their marbles in the same bag, what will be the dominance of the marbles (surplus of black ones or surplus of white ones)? How large will this surplus be?

This situation can be represented thus:

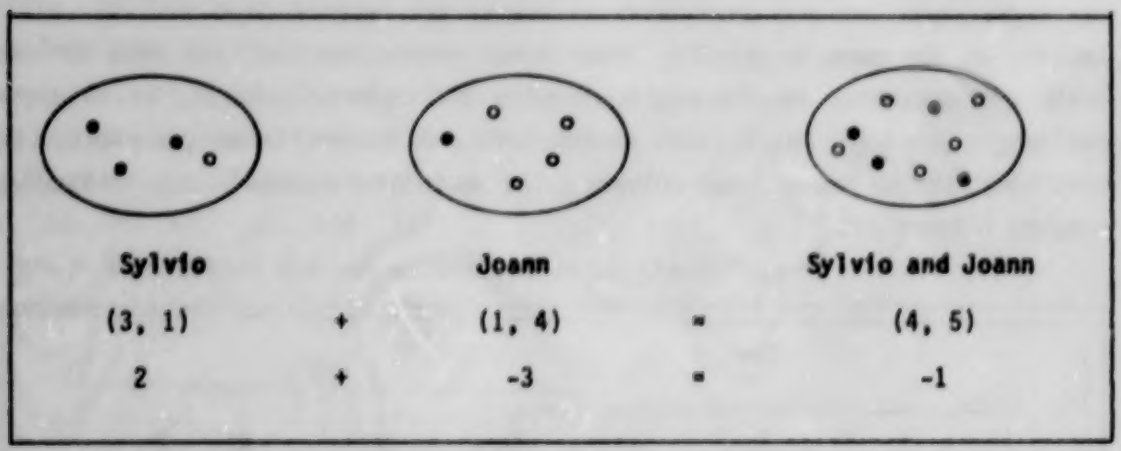


FIGURE 1

Note: Clearly, if either Silvio or Joann adds or subtracts an equal number of black and white marbles, the results will not change. Thus we could have:

$$(2, 0) + (2, 5) = (4, 5)$$
$$(6, 4) + (0, 3) = (6, 7)$$
$$\text{etc.}$$

since in terms of dominance these situations would all be expressed as in Figure 1:

$$2 + (-3) = -1$$

2.2.2 Black Tokens and White Tokens

Tokens of different colours may be used in place of the marbles described in the preceding section; the observations concerning the marbles are equally valid for the tokens.

Here are two examples of the use of this material:

EXAMPLE 1: Two pupils, Ian and Sylvia are seated face to face. The table separating them is divided as in Figure 2. Ian and Sylvia each have a number of black tokens and white tokens.

Ian begins the game by placing three black tokens on the left side of the table and one white on the right, forming the ordered pair (3, 1). Sylvia decides to place one black token on the left and four white on the right. By combining their tokens, by columns, Ian and Sylvia obtain the following results (Figure 2):

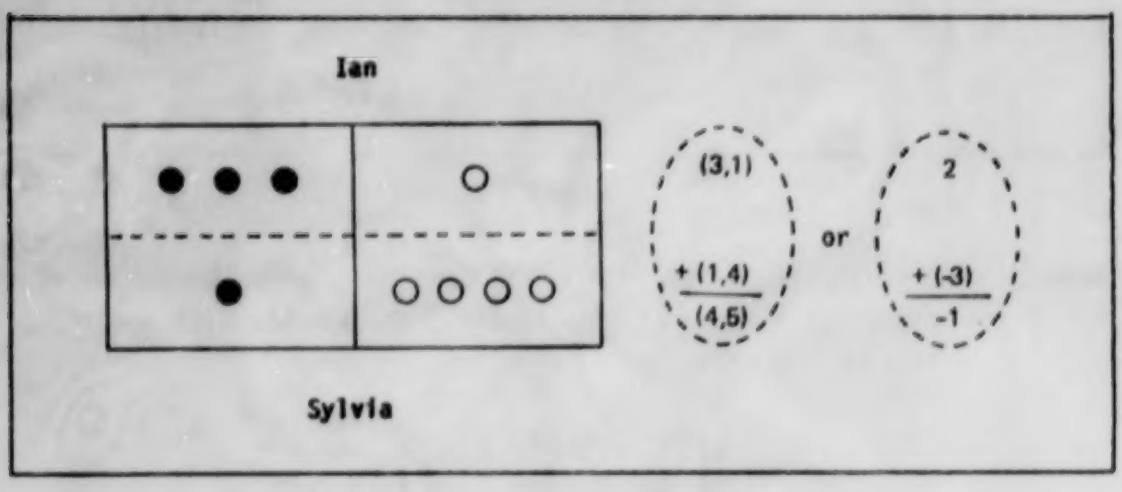


FIGURE 2

EXAMPLE 2: Now Ian arranges his tokens in the following manner:

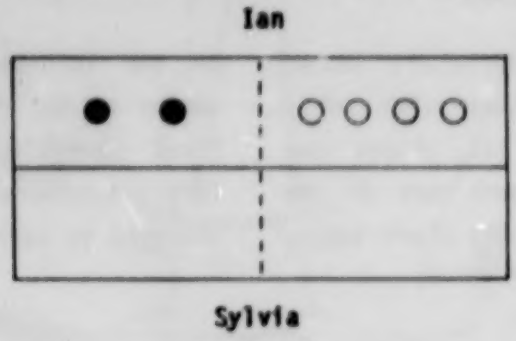


FIGURE 3

Sylvia, in turn must arrange the tokens so that there are the same number on the left side of the table as on the right. If she chooses to have five tokens on each half (See Figure 4), she must place three tokens on one side and one on the other, forming the ordered pair (3, 1). (Sylvia could just as easily have arranged her tokens so as to form ordered pairs such as (4, 2), (5, 3), (6, 4), ..., (44, 42), ..., (103, 101), ..., (474, 472), etc.). Figure 4 illustrates the sum which is thus obtained.

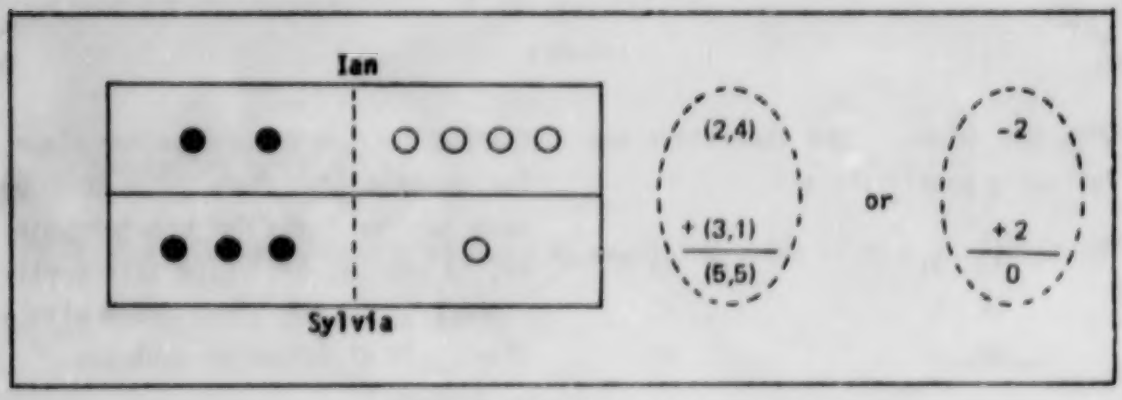


FIGURE 4

The addition of additive inverses is seen in the last example:

$$(2, 4) + (3, 1) = (5, 5)$$

or

$$-2 + 2 = 0$$

For all of these cases the comment in Section 2.2.1 remains valid, as it will be for Section 2.2.3.

2.2.3 Using the Balance

EXAMPLE 1: Two children are seated face-to-face on either side of a balance. The first, Ian, places some of his tokens on each pan of the balance: for example, 2 black tokens on the left-hand pan and 4 white tokens on the right-hand pan. After first observing that Ian's tokens form the ordered pair (2, 4), Sylvia attempts to restore equilibrium

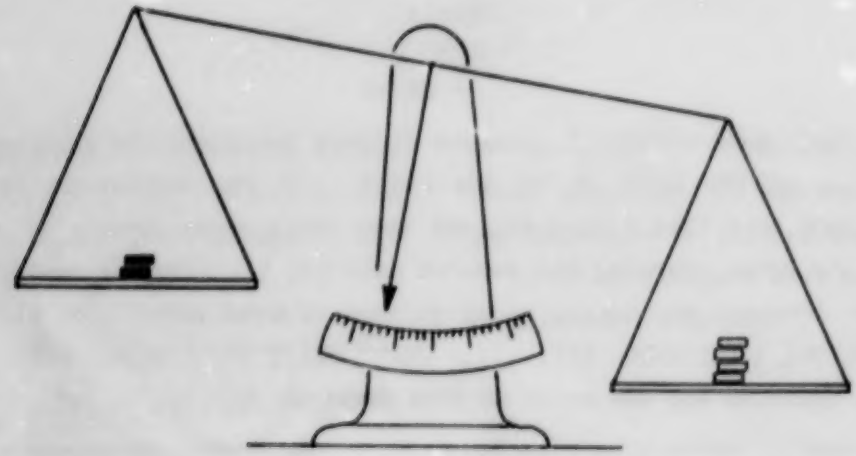


FIGURE 5

with her tokens. She then finds the following possibilities:

(5, 3)	(3, 1)
(6, 4)	(10, 8)
(7, 5)	(2, 0)

Finally she tries to find among these ordered pairs the one which is expressed in the simplest terms.

(28, 26) ⟶ (2, 0)

EXAMPLE 2: Ian can thereafter place, for example, two sets of tokens on each pan to form the ordered pairs (2, 4) and (6, 5), while Sylvia will attempt to restore equilibrium with a single set of tokens on each pan.

FIGURE 6

The arrangements corresponding to the ordered pairs (6, 5), (3, 2), or (2, 1) can restore equilibrium. Ian could in fact observe that:

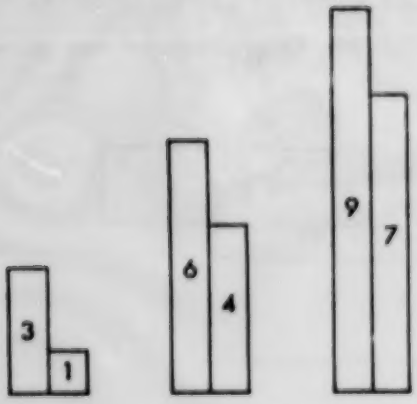
(2, 4) + (6, 5) + □ ⟶ (0, 0)

and find (6, 5), (3, 2), (2, 1), etc. as an answer. After a number of these exercises, it would be possible and desirable to explore the addition of relative integers written in a more conventional way:

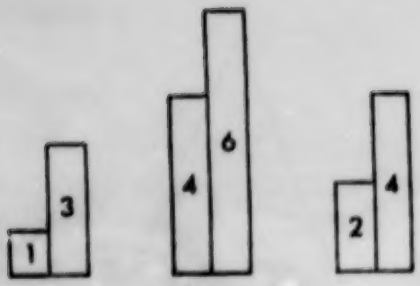
$$-2 + 1 + \square = 0$$

2.2.4 Cuisenaire Rods

(a) Exploration of relative integers by examining pairs of rods.



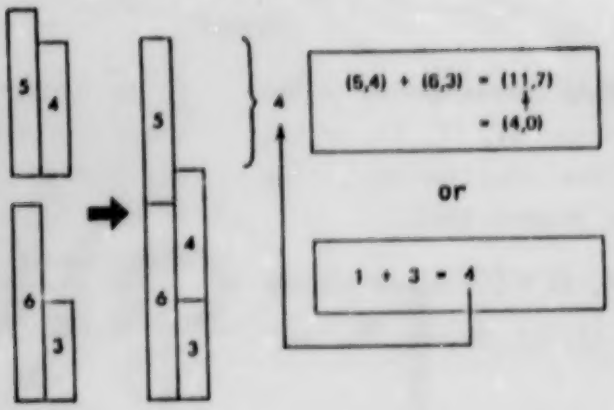
How are these pairs of rods alike?



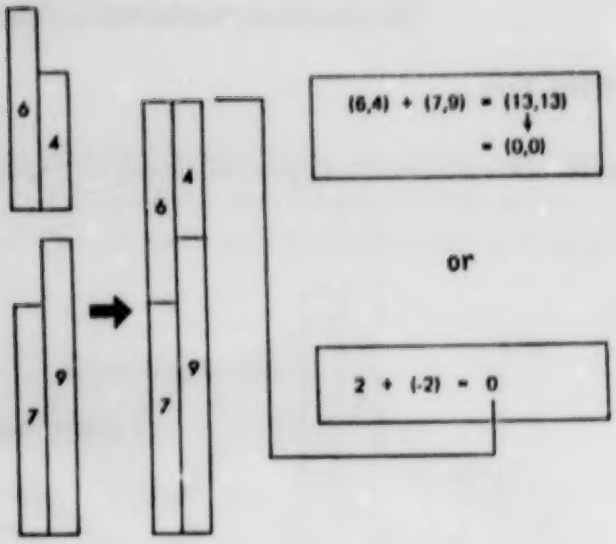
And these pairs? What difference is there with the previous groupings?

(b) Addition of relative integers using rods.

EXAMPLE 1:



EXAMPLE 2:

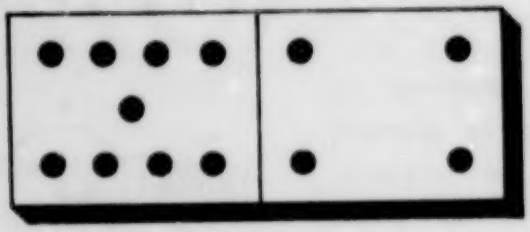


2.2.5 Dominoes

It is possible to build series of dominoes with sets of dots corresponding to the following ordered pairs:

1. (0, 0), (0, 1), (0, 2), ... (0, 9)
2. (1, 0), (1, 1), (1, 2), ... (1, 9)
3. (2, 0), ...
4. .
   .
   .
9. (9, 0), (9, 1) ... (9, 9)

Here, as an example, is the domino (9, 4)



The game consists of placing another domino below the first, so that the total number of dots on the left is equal to the number on the right. Thus, the following sum is arrived at through the operation depicted in Figure 7.

(9, 4) + (3, 8) = (12, 12)

or

5 + (-5) = 0

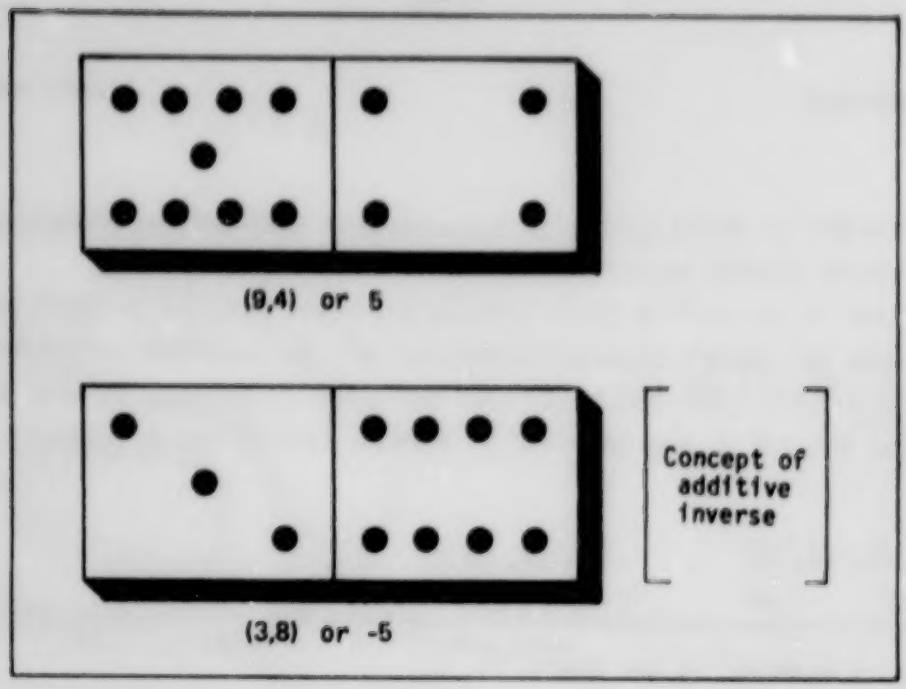
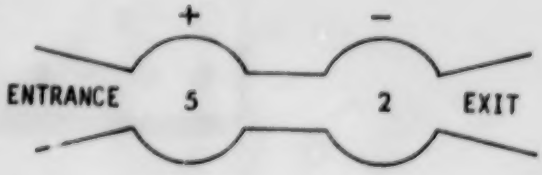


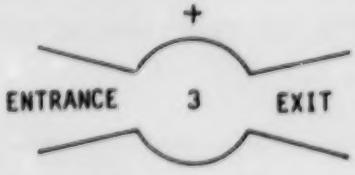
FIGURE 7

2.3 Function Machines

The use of function machines is another interesting way to approach the addition of relative integers. In fact, if we order the operations "add 5" and "subtract 2" the function machine might look like this:



Thus any number which passes through the machine undergoes two transformations, which, in reality, are equivalent to a single one: "add 3."



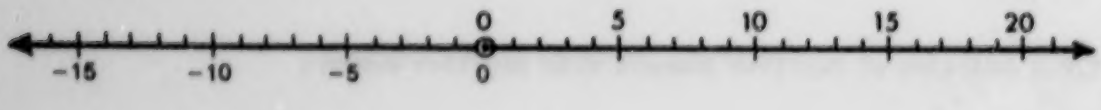
It is therefore relatively easy to conclude that the addition of 5 and of -2 results in 3:

$$5 + (-2) = 3$$

2.4 Number Line

2.4.1 The Addition of Negative Integers

The set of negative integers along a graduated straight line can be considered as the reflection or image of the positive integers,[1] where the zero would become the axis of symmetry of these two sets along the line. The addition of negative integers (shown in red) is done in the same way as for the positive integers



(shown in black) using the same operations.

Thus,

7 + 13 = 20 (positive numbers)
5 + 8 = 13 (negative numbers)

It is no more difficult to add negative numbers to each other than it is to add positive numbers.

2.4.2 Additive Inverse

The situation becomes more complicated, however, when positive numbers are to be added to negative numbers. It is here that the concept of <u>additive inverse</u> takes on full importance.

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[1] See the Introduction, page 1.

It is possible to write:

4	+	4	=	0
7	+	7	=	0
121	+	121	=	0

where the numbers

4	et	4
7	et	7
121	et	121

are each other's respective additive inverses.

2.4.3 Addition of Positive Numbers with Negative Numbers

In order to add 13 + 5, for example, the properties made explicit in 2.4.1 and 2.4.2 are used:

13 + 5

becomes	(8 + 5) + 5	(addition of positive integers)
or	8 + (5 + 5)	(additive inverses)
or	8 + 0	
or	8	

Chapter 3

Subtraction of Relative Integers

With the set $\mathbb{N}$, the addition and multiplication of numbers always gives answers which are elements of this same set. However, this is not true for certain subtractions such as $4 - 7 = \square$, or $6 - 11 = \square$.

The extension of the set of natural numbers to the larger set of relative integers corrects this situation since for each pair of relative integers there can be, through this same subtraction operation, another corresponding relative integer. The set $\mathbb{N}$ allowed only addition and multiplication; the set $\mathbb{Z}$ allows subtraction as well:

$4 - 7 = -3$ $\qquad$ $6 - 11 = -5$

Helping children understand how to obtain these results, however, is not a particularly easy task.

The teacher may often experience difficulty imagining simple situations which might lead the child to perform such operations.

It would perhaps be better to help the child understand more clearly the links to be established between the operations of addition and subtraction.[1]

Thus, $5 - 2 = 3$ or $5 - 3 = 2$ because $2 + 3 = 5$. Similarly, it is possible to write $4 - (-1) = 5$ since $-1 + 5 = 4$ (See Chapter 2). This is far from obvious. It would therefore be wise in elementary school classes always to accompany a subtraction of relative integers with appropriate manipulations. The material used in the learning of addition in Chapter 2 may prove useful.

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1 See Booklet C of the Curriculum Guide, Chapter 3

Note the following addition:[1]

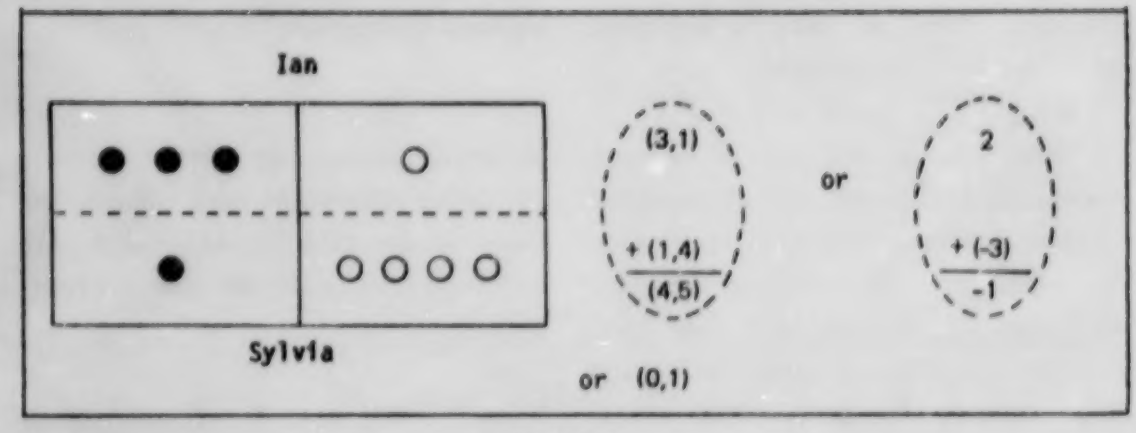

FIGURE 1

The corresponding subtractions would be:

(1) (4, 5) - (3, 1) = (1, 4) -1 - 2 = -3
or (2) (4, 5) - (1, 4) = (3, 1) -1 - (-3) = 2

(See Figures 2 and 3)

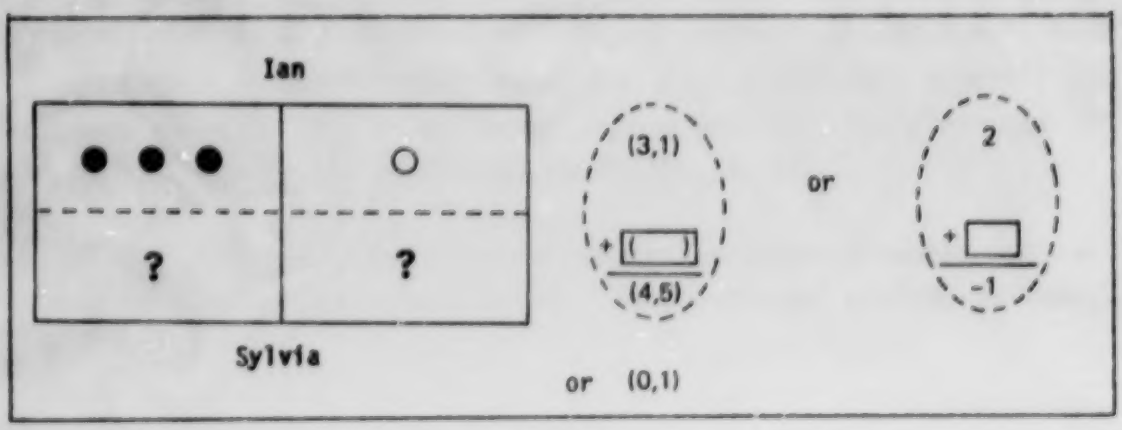

FIGURE 2

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[1] See Example 1, Section 2.2.2

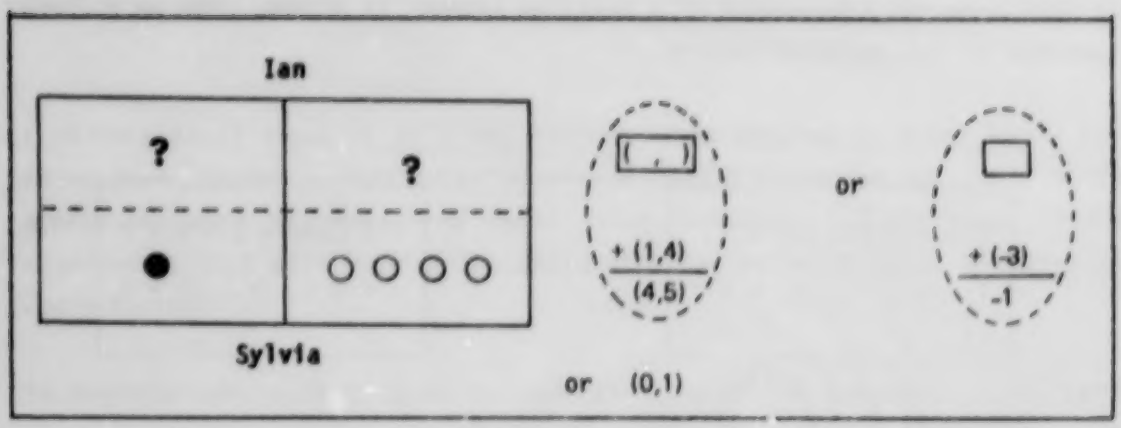

FIGURE 3

This procedure does not present major problems. A problem would arise, however, if the ordered pair (4, 5) for the sum is replaced by the equivalent ordered pair: (0, 1).

The procedure to be followed in this case is to transform such an ordered pair into another equivalent ordered pair in which the terms are equal to or larger than those of the ordered pair to be subtracted.

Thus, in order to subtract, for example, the ordered pairs (3, 1) or (1, 4) from the ordered pair (0, 1) the latter ordered pair is first transformed to an equivalent ordered pair such as (3, 4), (4, 5), (5, 6), ...

Other analogous examples drawn from the various situations presented in Chapter 2 on addition can easily be imagined: the use of sets, balance, dominoes, Cuisenaire rods.

Note: If need be, it would be possible to use simply a definition of subtraction based on the use of the additive inverse.

In this case the subtraction of a relative integer is accomplished through the addition of its additive inverse.

$3 - 4 = 3 + (4) = 7$ $7 - 3 = 7 + 3 = 4$	or	$3 - (-4) = 3 + 4 = 7$ $7 - 3 = 7 + (-3) = 4$

Obviously, the concept of additive inverse is not easy to grasp even if it is an essential element of the latter approach. For this reason it is generally agreed to introduce this aspect of the subtraction of relative integers only at the secondary level.

CONCLUSION

A child's first exposure to a new subject often results in fixed impressions about that subject. Particular attention should therefore be given to the approaches used in presenting these first concepts. It is these initial experiences that will provide the child with the basis on which to build and learn further.

To motivate and to facilitate the understanding of the concepts, it is important to provide learning situations drawn from the child's daily life. The situations cited as examples in the introduction can always be used for this purpose. Teachers are encouraged to develop other problems which will fall within the realm of the child's experiences.

For example, the pupils might be asked what will happen if the mercury in the thermometer falls below the zero mark. They might also be asked what can be done with an equation of the form

$$4 = 7 + \square \ .$$

The pupils are not expected to be able to answer these questions from the start, but only to realize that it is necessary to "invent" new numbers in order to solve certain types of problems which are very common.

APPENDIX I

Relative Integers and "Patterns"

Attempting to complete a series according to a model roughed out in the first elements of the series can also provide an opportunity to consider the relative integers as an extension of the natural numbers. (See page 3, 1st paragraph, Section 1.1). The use of a calculator in class can easily result in such situations.

Here are some examples:

EXAMPLE 1:
5 - 1 = 4
5 - 2 = 3
5 - 3 = 2
5 - 4 = 1
5 - 5 = ...
5 - 6 = ...
5 - 7 = ...

EXAMPLE 2:
8 + 3 = 11
8 + 2 = 10
8 + 1 = 9
8 + 0 = 8
8 + (-1) = 7
.....

EXAMPLE 3:
6 - 3 = 3
6 - 2 = 4
6 - 1 = 5
6 - 0 = ...
6 - (-1) = ...
6 - (-2) = ...
6 - (-3) = ...
.....

EXAMPLE 4:
4 - 1 = 3
3 - 1 = 2
2 - 1 = 1
1 - 1 = ...
0 - 1 = ...
-1 - 1 = ...
-2 - 1 = ...
.....

EXAMPLE 5:
-2 + (-3) = -5
-2 + (-1) = -3
-2 + 1 = -1
-2 + 3 = 1
.....

Even if some of the results obtained appear surprising to the child, he should be able to realize that they are part of a continuing process which is, to all appearances, acceptable to him.

APPENDIX II

Proportion of Time to Devote to the Teaching of Relative Integers